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On Numerical Experiments with Symmetric Semigroups Generated by Three Elements and Their Generalization

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Abstract

We give a simple explanation of numerical experiments of V. Arnold with two sequences of symmetric numerical semigroups, $\mathsf{S}(4, 6+4k, 87-4k)$ and $\mathsf{S}(9, 3+9k, 85-9k)$ generated by three elements. We present a generalization of these sequences by numerical semigroups $\mathsf{S}(r_1^2, r_1 r_2 + r_1^2 k, r_3 - r_1^2 k)$, $k \in \mathbb{Z}$, $r_1, r_2, r_3 \in \mathbb{Z}^+$, $r_1 \geq 2$ and $\gcd(r_1, r_2) = \gcd(r_1, r_3) = 1$, and calculate their universal Frobenius number $\Phi(r_1, r_2, r_3)$ for the wide range of k providing semigroups be symmetric. We show that this kind of semigroups admit also nonsymmetric representatives. We describe the reduction of the minimal generating sets of these semigroups up to $\{r_1^2, r_3 - r_1^2 k\}$ for sporadic values of k and find these values by solving the quadratic Diophantine equation.

Keywords: Symmetric numerical semigroups, Frobenius number

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1 Introduction

In experiments with Frobenius numbers $F(d_1, d_2, d_3)$ of numerical semigroups, generated by a tuple of three elements $\{d_1, d_2, d_3\}$, V. Arnold has mentioned two strange arithmetic facts [1] (see [1], Remark 1),

$$F(4, 6+4k, 87-4k) = 89 , \quad k = 0, 1, \ldots, 14,\ k \neq 8 , \tag{1}$$

$$F(9, 3+9k, 85-9k) = 167 , \quad k = 1, \ldots, 7 . \tag{2}$$

[1] Throughout the paper we use the term *Frobenius number* whose standard definition dates back to G. Frobenius, I. Schur and A. Brauer [2] and denotes the largest integer that is not representable as a linear combination with nonnegative integer coefficients of a given tuple of positive integers $\{d_1, \ldots, d_m\}$, $\gcd(d_1, \ldots, d_m) = 1$. V. Arnold [1] had used a different definition of this term, so in (1) and (2) he got numbers 90 and 168 instead of 89 and 167.

In sections 2 and 3 we give a simple proof of these statements. In fact, we prove

$$F(4,6+4k,87-4k)=89\ ,\quad -1\le k\le 14\ ;\quad F(9,3+9k,85-9k)=167\ ,\quad 0\le k\le 7\ . \qquad (3)$$

The proof is based on observation that two sequences of triples,

$$\{4,6+4k,87-4k\},\ 0\le k\le 14\ ,\quad \text{and}\quad \{9,3+9k,85-9k\},\ 1\le k\le 6\ , \qquad (4)$$

present the generators of symmetric numerical semigroups generated by three elements. The case $k=-1$ in the 1st triple and the cases $k=0$, $k=7$ in the 2nd triple are special and reduce the semigroups, which are generated by three elements, up to symmetric semigroups, which are generated by two elements. In sections 4 and 5 we generalize both examples (4) to most generic triple and analyze its associated symmetric semigroups. In section 6 we discuss a case when this generating triple is reduced up to generating pair, and values of their elements are coming by finding the integer points in plane algebraic curve of degree 2.

1.1 Basic Facts on Numerical Semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$

Following [3] we recall basic definitions and known facts on algebra of the numerical semigroups generated by m elements which are necessary here, and focus on their symmetric subsets. For short we denote the generating tuple $(d_1,\ldots,d_m)$ by $\mathbf{d}^m$.

A semigroup $\mathsf{S}\left(\mathbf{d}^m\right)=\{s\in\mathbb{Z}^+\cup\{0\}\mid s=\sum_{i=1}^m x_i d_i,\ x_i\in\mathbb{Z}^+\cup\{0\}\}$ is said to be generated by minimal set of m natural numbers,

$$\gcd(d_1,\ldots,d_m)=1\ ,\quad \min(d_1,\ldots,d_m)\ge m\ , \qquad (5)$$

if neither of its elements is linearly representable by the rest of elements. Throughout the paper we call such semigroups m-dimensional (mD). Denote by $\Delta\left(\mathbf{d}^m\right)$ the complement of $\mathsf{S}\left(\mathbf{d}^m\right)$ in $\mathbb{Z}^+$, i.e. $\Delta\left(\mathbf{d}^m\right)=\mathbb{Z}^+\setminus\mathsf{S}\left(\mathbf{d}^m\right)$ and call it a set of gaps. The Frobenius number of semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ is defined as follows,

$$F\left(\mathbf{d}^m\right)=\max\Delta\left(\mathbf{d}^m\right)\ . \qquad (6)$$

A semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ is called symmetric if for any integer s the following condition holds: iff $s\in\mathsf{S}\left(\mathbf{d}^m\right)$ then $F\left(\mathbf{d}^m\right)-s\notin\mathsf{S}\left(\mathbf{d}^m\right)$. Otherwise $\mathsf{S}\left(\mathbf{d}^m\right)$ is called nonsymmetric. Notably that all semigroups $\mathsf{S}\left(d_1,d_2\right)$, $\min(d_1,d_2)\ge 2$, generated by two elements, i.e. two-dimensional (2D) semigroups, are symmetric. Combining the last fact with the early statement of Watanabe on the symmetric semigroups of dimension $m\ge 3$ (see [5], Lemma 1) we come to important statement.

Lemma 1 *Let* $\mathsf{S}(c_1, c_2)$ *be a numerical semigroup,* a *and* b *be positive integers,* $\gcd(a, b) = 1$*. If* $a \in \mathsf{S}(c_1, c_2)$*, then the semigroup* $\mathsf{S}(bc_1, bc_2, a)$ *is symmetric.*

Remark 1 *Note that a requirement* $a \in \mathsf{S}(c_1, c_2)$ *can be provided in two ways. First, it is satisfied if the Frobenius number of 2D semigroup* $\mathsf{S}(c_1, c_2)$ *is exceeded by the third generator 'a',*

$$a \geq C(c_1, c_2) = 1 + F(c_1, c_2) = (c_1 - 1)(c_2 - 1) \ , \tag{7}$$

where $C(c_1, c_2)$ *denotes a conductor of semigroup* $\mathsf{S}(c_1, c_2)$*. The last equality in (7) comes due to the known Sylvester formula [4]. In the case* $a < (c_1 - 1)(c_2 - 1)$ *there is another way to provide the containment* $a \in \mathsf{S}(c_1, c_2)$*, namely, to have the number 'a' among the nongaps of semigroup* $\mathsf{S}(c_1, c_2)$*, i.e.* $a \in \mathsf{S}(c_1, c_2) \cap [0, F(c_1, c_2)]$*. The last requirement is much harder to verify than (7).*

The last case will be also observed in further calculation (see section 3, a case $k = 6$ for the triple $\{9, 3 + 9k, 85 - 9k\}$). A powerful tool to study the symmetric numerical semigroups is the Herzog formula [4] for the Frobenius number (for details see section 6.2 in [3]). Being adapted for symmetric semigroup $\mathsf{S}(bc_1, bc_2, a)$ in Lemma 1, it looks as follows,

$$F(bc_1, bc_2, a) = bc_1c_2 + ab - (bc_1 + bc_2 + a) \ . \tag{8}$$

Keeping in mind Lemma 1 consider in details the two sequences of triples given in (4).

2 Symmetric Semigroups $\mathsf{S}(4, 6 + 4k, 87 - 4k)$

The triple $\{4, 6 + 4k, 87 - 4k\}$ has always two relative non-prime generators bc_1 and bc_2, namely, $c_1 = 2$, $c_2 = 3 + 2k$ and $b = 2$. In order to satisfy Lemma 1 we have to provide the containment $87-4k \in \mathsf{S}(2, 3+2k)$. By (5) the requirement $3+2k \geq 2$ brings us to the lower bound for k, $k \geq 0$. The upper bound comes by another claim for conductor $C(2, 3 + 2k)$ of semigroup $\mathsf{S}(2, 3 + 2k)$,

$$87 - 4k \geq C(2, 3 + 2k) = 2 \cdot (3 + 2k) - (2 + 3 + 2k) + 1 \quad \rightarrow \quad 85 \geq 6k \quad \rightarrow \quad k \leq 14 \ . \tag{9}$$

Thus, by Lemma 1 the numerical semigroups $\mathsf{S}(4, 6 + 4k, 87 - 4k)$, $0 \leq k \leq 14$, are symmetric. Applying (8) we get

$$F(4, 6 + 4k, 87 - 4k) = 4 \cdot (3 + 2k) + 2 \cdot (87 - 4k) - (4 + 6 + 4k + 87 - 4k) = 89 \ . \tag{10}$$

The higher values of k are bounded by the claim (5): $87 - 4k \geq 3$ that gives $k \leq 21$. The corresponding semigroups $\mathsf{S}(4, 6 + 4k, 87 - 4k)$, $15 \leq k \leq 21$, are isomorphic to the 2D symmetric

semigroups:

$$\mathsf{S}(4,66,27)=\mathsf{S}(4,27),\;\;\mathsf{S}(4,70,23)=\mathsf{S}(4,23),\;\;\mathsf{S}(4,74,19)=\mathsf{S}(4,19),\;\;\mathsf{S}(4,78,15)=\mathsf{S}(4,15),$$
$$\mathsf{S}(4,82,11)=\mathsf{S}(4,11),\;\;\mathsf{S}(4,86,7)=\mathsf{S}(4,7),\;\;\mathsf{S}(4,90,3)=\mathsf{S}(4,3)\;.$$

Their Frobenius numbers can be found by Sylvester formula,

$$F(4,66,27)=77,\;\;F(4,70,23)=65,\;\;F(4,74,19)=53,\;\;F(4,78,15)=41,$$
$$F(4,82,11)=29,\;\;F(4,86,7)=17,\;\;F(4,90,3)=5\;.$$

The case $k=-1$ is a special one. It corresponds to semigroup $\mathsf{S}(4,2,91)$ with non-minimal generating set $\{4,2,91\}$. It can be reduced up to $\{2,91\}$ which generates a semigroup $\mathsf{S}(2,91)$. The Frobenius number of the latter semigroup follows by Sylvester formula, $F(2,91)=89$.

3 Symmetric Semigroups $\mathsf{S}(9,3+9k,85-9k)$

The triple $\{9,3+9k,85-9k\}$ has always two relative non-prime generators bc_1 and bc_2, namely, $c_1=3$, $c_2=3k+1$ and $b=3$. In order to satisfy Lemma 1 we have to provide the containment $85-9k\in\mathsf{S}(3,3k+1)$. By (5) the requirement $3k+1\geq 2$ brings us to the lower bound for k, $k\geq 1$. The upper bound comes by another claim for conductor $C(3,3k+1)$ of semigroup $\mathsf{S}(3,3k+1)$,

$$85-9k\geq C(3,3k+1)=3\cdot(3k+1)-(3+3k+1)+1\;\;\rightarrow\;\;85\geq 15k\;\;\rightarrow\;\;k\leq 5\;. \tag{11}$$

Thus, by Lemma 1 the numerical semigroups $\mathsf{S}(9,3+9k,85-9k)$, $1\leq k\leq 5$, are symmetric. Applying (8) we get

$$F(9,3+9k,85-9k)=9\cdot(3k+1)+3\cdot(85-9k)-(9+9k+3+85-9k)=167\;. \tag{12}$$

A case $k=6$ gives rise to another symmetric semigroup $\mathsf{S}(9,57,31)$ which satisfies Lemma 1: $31\in\mathsf{S}(3,19)$, however $31<C(3,19)$. Making use of (12) we get $F(9,57,31)=167$.

Finally, two other cases $k=0$ and $k=7$ give rise to 3D semigroups $\mathsf{S}(9,3,85)$ and $\mathsf{S}(9,66,22)$ with non-minimal generating sets $\{9,3,85\}$ and $\{9,66,22\}$, respectively. However, they can be reduced up to the 2D semigroups $\mathsf{S}(3,85)$ and $\mathsf{S}(9,22)$, respectively. The Frobenius numbers of the two last semigroups follow by Sylvester formula, $F(9,3,85)=167$ and $F(9,66,22)=167$.

The higher values of k are bounded by the claim (5): $85-9k\geq 3$ that gives $k\leq 9$. The corresponding semigroups $\mathsf{S}(9,3+9k,85-9k)$, $k=8,9$, are isomorphic to the 2D symmetric semigroups:

$$\mathsf{S}(9,75,13)=\mathsf{S}(9,13)\;,\;\;\mathsf{S}(9,84,4)=\mathsf{S}(9,4)\;.$$

Their Frobenius numbers follow by Sylvester formula, $F(9,75,13)=95,\ F(9,84,4)=23$.

It is worth to mention that in the whole range of varying parameter k with the values of the triples' elements exceeding 1 both sequences of these triples in sections 2 and 3 give rise only to symmetric semigroups either three-dimensional or two-dimensional,

$$\{4,6+4k,87-4k\},\ -1\le k\le 21\ ,\quad \text{and}\quad \{9,3+9k,85-9k\},\ 0\le k\le 9\ . \tag{13}$$

This observation is important not less than the claim (3) on universality of the Frobenius numbers 89 and 167. However the range of application of (13) is much wider than (4).

4 Numerical Semigroups $\mathsf{S}(r_1^2, r_1r_2+r_1^2k, r_3-r_1^2k)$

In this section we generalize both examples discussed by V. Arnold in [1]. For the first glance, a most generic triple is of the form,

$$\left\{u^2v^2, u^2vw+u^2v^2k, t-u^2v^2k\right\},\left\{\begin{array}{l} k\in\mathbb{Z},\\ u,v,w,t\in\mathbb{Z}^+\end{array}\right.,\left\{\begin{array}{l}\gcd(u,v)=\gcd(u,w)=\gcd(v,w)=1,\\ \gcd(u,t)=\gcd(v,t)=1,\quad uv\ge 2.\end{array}\right.$$

However, by comparison with Arnold's examples, the last triple has one serious lack. Indeed, consider a symmetric semigroup $\mathsf{S}\left(u^2v^2, u^2vw+u^2v^2k, t-u^2v^2k\right)$ and calculate by formula (8) its Frobenius number,

$$F(u,v,w,t,k)=(t+u^2vw)(v-1)-u^2v^2+ku^2v^3(1-u^2)\ . \tag{14}$$

In contrast to examples in [1], an expression in (14) is dependent on k. This dependence disappears iff $u=1$. The generating triples of only such kind will be a subject of interest in this article. Henceforth, denote $v=r_1,\ w=r_2,\ t=r_3$ and consider a triple which is governed by three parameters, r_1, r_2 and r_3,

$$\left\{r_1^2, r_1r_2+r_1^2k, r_3-r_1^2k\right\},\ \ k\in\mathbb{Z},\ r_1,r_2,r_3\in\mathbb{Z}^+,\ r_1\ge 2\ \text{ and }\ \gcd(r_1,r_2)=\gcd(r_1,r_3)=1. \tag{15}$$

In new notations r_1, r_2 and r_3 and by $u=1$ formula (14) reads

$$\Phi\left(r_1,r_2,r_3\right)=(r_1-1)(r_1r_2+r_3)-r_1^2\ . \tag{16}$$

There are two different ways to symmetrize the 3D numerical semigroup $\mathsf{S}(r_1^2, r_1r_2+r_1^2k, r_3-r_1^2k)$. The 1st way is to choose k such that the necessary conditions in Lemma 1 be satisfied. The 2nd way is to choose k such that the generating triple is non-minimal, i.e. one of its elements is linearly representable by the rest of elements. In other words, one can arrive at symmetric semigroup preserving the dimension 3 of generic semigroup or reducing it by 1.

Unfortunately, a complete analysis of symmetrization of numerical semigroup generated by the triple (15) encounter a serious difficulty in both ways of its performing. This is related to non-analytic nature of both containments $r_3-r_1^2\bar{k}\in\mathsf{S}(r_1,r_2+r_1\bar{k})$ and $r_1r_2+r_1^2\tilde{k}\in\mathsf{S}(r_1^2,r_3-r_1^2\tilde{k})$. In other words, one cannot write the explicit formulas of $\bar{k}$ and $\tilde{k}$ via r_1,r_2,r_3 for the whole set of nongaps for both semigroups $\mathsf{S}(r_1,r_2+r_1\bar{k})$ and $\mathsf{S}(r_1^2,r_3-r_1^2\tilde{k})$. What we can do only to make use of (7) providing the ranges of $\bar{k}$ and $\tilde{k}$ when the elements $r_3-r_1^2\bar{k}$ and $r_1r_2+r_1^2\tilde{k}$ exceed the Frobenius numbers of corresponding semigroups, respectively. According to Remark 1 this symmetrizes an initial semigroup $\mathsf{S}(r_1^2,r_1r_2+r_1^2k,r_3-r_1^2k)$ at $k=\bar{k},\tilde{k}$.

In section 4.1 we find the range of k-values for the sequence of symmetric semigroups generated by the triple (15) with equal Frobenius numbers (16). In section 4.2 we give an affirmative answer to another question: whether the sequence (15) does contain also a triple associated with nonsymmetric semigroups.

4.1 Symmetric semigroups and special values of k

We start with the 1st way of symmetrization and assume that $r_1k+r_2\neq 1$. By Lemma 1 a numerical semigroup $\mathsf{S}(r_1^2,r_1r_2+r_1^2k,r_3-r_1^2k)$ is symmetric if the following containment holds, $r_3-r_1^2k\in\mathsf{S}(r_1,r_1k+r_2)$. By (5) it brings us necessarily to the two inequalities imposed onto generators,

$$r_1k+r_2\geq 2\ ,\quad r_3-r_1^2k\geq 3\ . \tag{17}$$

Denote two special values of k,

$$k_1=\frac{2-r_2}{r_1}\ ,\quad k_2=\frac{r_3-3}{r_1^2}\ , \tag{18}$$

and find a range of k where both inequalities (17) do not contradict each other,

$$\text{if}\quad k_1\leq k_2\quad\text{then}\quad k_1\leq k\leq k_2\ . \tag{19}$$

On the other hand,

$$\text{if}\quad k_1>k_2\quad\text{or}\quad k\leq k_1\quad\text{or}\quad k\geq k_2\ , \tag{20}$$

then the corresponding k does not provide the necessary requirement (5).

Apply Lemma 1 to semigroup $\mathsf{S}(r_1^2,r_1r_2+r_1^2k,r_3-r_1^2k)$. Being 2-dimensional, the symmetric semigroup $\mathsf{S}(r_1,r_1k+r_2)$ is associated with Frobenius number according to Sylvester formula. Following (7) write an inequality

$$r_3-r_1^2k\geq 1+F(r_1,r_1k+r_2)=(r_1-1)(r_1k+r_2-1)\ . \tag{21}$$

It gives rise to another special value of k,

$$k \leq k_3 \;, \qquad k_3 = \frac{r_3 - (r_1 - 1)(r_2 - 1)}{r_1(2r_1 - 1)} \;, \tag{22}$$

where our 3D semigroup is symmetric. If the inequality (21) is broken,

$$r_3 - r_1^2 k \leq F(r_1, r_1 k + r_2) \;, \qquad \text{or} \qquad k \geq k_3 + \frac{1}{r_1(2r_1 - 1)} \;, \tag{23}$$

then a containment $r_3 - r_1^2 k \in \mathsf{S}(r_1, r_1 k + r_2)$ can be still provided if $r_3 - r_1^2 k$ is a nongap of semigroup $\mathsf{S}(r_1, r_1 k + r_2)$. Note that inequality (23) admits also the existence of nonsymmetric semigroups generated by triple (15) if $r_3 - r_1^2 k$ is a gap of $\mathsf{S}(r_1, r_1 k + r_2)$.

Let us find the common range of k which is consistent with (19), (20), (22) and (23) and dependent on interrelationships between k_1, k_2 and k_3. By comparison of expressions (18) and (22) for k_1, k_2 and k_3 we find the constraints when these relationships are valid. Below we list these relationships presented in terms of r_1, r_2 and r_3.

$$k_1 \leq k_2 \leq k_3 \;, \qquad \text{or} \qquad 2r_1 + 3 \leq r_1 r_2 + r_3 \leq \frac{r_1^2 + 5r_1 - 3}{r_1 - 1} \;, \tag{24}$$

$$k_1 \leq k_3 \leq k_2 \;, \qquad \text{or} \qquad \frac{r_1^2 + 5r_1 - 3}{r_1 - 1} \leq r_1 r_2 + r_3 \;, \quad 3r_1 - 1 \leq r_1 r_2 + r_3 \;, \tag{25}$$

$$k_3 \leq k_1 \leq k_2 \;, \qquad \text{or} \qquad 2r_1 + 3 \leq r_1 r_2 + r_3 \leq 3r_1 - 1 \;. \tag{26}$$

4.1.1 Semigroup's reduction: $\mathsf{S}(r_1^2, r_1 r_2 + r_1^2 k, r_3 - r_1^2 k) \;\rightarrow\; \mathsf{S}(r_1, r_3 - r_1^2 k)$

Consider the case $r_1 k + r_2 = 1$. Indeed, by this relation the two first generators of the triple (15) become linearly dependent, and therefore the 3D numerical semigroup is reduced up to the 2D semigroup $\mathsf{S}(r_1, r_3 - r_1^2 k)$ which is always symmetric. Summarizing these requirements we conclude that a numerical semigroup $\mathsf{S}(r_1^2, r_1 r_2 + r_1^2 k_4, r_3 - r_1^2 k_4)$ is symmetric if

$$k_4 = \frac{1 - r_2}{r_1} \in \mathbb{Z} \;. \tag{27}$$

The corresponding generator $r_3 - r_1^2 k_4$ and the Frobenius number $F(r_1, r_3 - r_1^2 k_4)$ read

$$r_3 - r_1^2 k_4 = r_3 + r_1 r_2 - r_1 \;, \quad F(r_1, r_3 - r_1^2 k_4) = (r_3 + r_1 r_2)(r_1 - 1) - r_1^2 \;. \tag{28}$$

Note that $k_4 = k_1 - 1/r_1$, i.e. $k_1 - k_4 \leq 1/2$. In fact, this expands the range (19) of existence of symmetric numerical semigroups generated by the triple (15) up to $k_4 \leq k \leq k_2$. Note that two Frobenius numbers $F(r_1, r_3 - r_1^2 k_4)$ and $\Phi(r_1, r_2, r_3)$ given by (28) and (16) coincide. If $r_2 = 1$ then there always exists 2D semigroup $\mathsf{S}(r_1, r_3)$ which comes by putting $k = 0$ into (15).

4.1.2 Semigroup's reduction: $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k) \;\to\; \mathsf{S}(r_1^2, r_3 - r_1^2k)$

Next, consider the case $r_1r_2 + r_1^2k \in \mathsf{S}(r_1^2, r_3 - r_1^2k)$ and find the value k_5 such that for all $k > k_5$ the above containment is provided. For this purpose, in accordance with (7), we have to satisfy the following inequality,

$$r_1r_2 + r_1^2k > F(r_1^2, r_3 - r_1^2k) = (r_3 - r_1^2k)(r_1^2 - 1) - r_1^2 \; . \tag{29}$$

It gives another special value of k,

$$k > k_5 \;, \qquad k_5 = \frac{(r_3-1)r_1^2 - (r_3 + r_1r_2)}{r_1^4} \; . \tag{30}$$

By (18) and (30) it follows

$$k_2 - k_5 = \frac{r_1r_2 + r_3 - 2r_1^2}{r_1^4} \;, \qquad \text{i.e.} \quad k_2 \ge k_5 \quad \text{iff} \quad r_1r_2 + r_3 \ge 2r_1^2 \; . \tag{31}$$

Find a value k_6 where the Frobenius number $F(r_1^2, r_3 - r_1^2k_6)$ coincides with $\Phi(r_1, r_2, r_3)$,

$$F(r_1^2, r_3 - r_1^2k_6) = (r_1 - 1)(r_1r_2 + r_3) - r_1^2 \quad \to \quad k_6 = \frac{r_3 - r_2}{r_1(r_1+1)} \; . \tag{32}$$

4.1.3 Semigroup's reduction: $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k) \;\to\; \mathsf{S}(r_1r_2 + r_1^2k, r_3 - r_1^2k)$

Finally, consider the case $r_1^2 \in \mathsf{S}(r_1r_2 + r_1^2k, r_3 - r_1^2k)$ and find the k-values such that the above containment is provided. In accordance with (7), we have to satisfy the following inequality,

$$r_1^2 \ge F(r_1r_2 + r_1^2k, r_3 - r_1^2k) + 1 \; . \tag{33}$$

It gives two other special values of k,

$$k \le k_7 \quad \text{or} \quad k_8 \le k \;, \quad \text{where} \quad k_8 - k_7 = \frac{\sqrt{(r_3 + r_1r_2 - 2)^2 - 4r_1^2}}{r_1^2} \;, \tag{34}$$

$$k_7 = \frac{r_3 - r_1r_2 - \sqrt{(r_3 + r_1r_2 - 2)^2 - 4r_1^2}}{2r_1^2} \;, \quad k_8 = \frac{r_3 - r_1r_2 + \sqrt{(r_3 + r_1r_2 - 2)^2 - 4r_1^2}}{2r_1^2} \; . \tag{35}$$

By (34) we have $k_8 \ge k_7$, if $r_3 + r_1r_2 \ge 2 + 2r_1$, otherwise an inequality (33) holds for any k. Making use of formulas (18), (35) and calculating two differences, $k_8 - k_2$ and $k_1 - k_7$, we get

$$a) \;\; k_8 \ge k_2 \quad \text{if} \quad r_3 + r_1r_2 \ge 4 + \frac{r_1^2}{2} \;; \qquad b) \;\; k_1 \ge k_7 \quad \text{if} \quad r_3 + r_1r_2 \ge \frac{5r_1^2 - 1}{2r_1 - 1} \; . \tag{36}$$

By comparison of criteria in (31) and (36) we obtain

$$\text{if} \quad k_2 \ge k_5 \;, \quad \text{then} \quad k_7 \le k_1 \quad \text{and} \quad k_2 \le k_8 \; . \tag{37}$$

In other words, if $r_1r_2+r_3\geq 2r_1^2$ then the semigroup's reduction $\mathsf{S}(r_1^2,r_1r_2+r_1^2k,r_3-r_1^2k)\to\mathsf{S}(r_1r_2+r_1^2k,r_3-r_1^2k)$ cannot be observed. On the other hand,

$$\text{if}\quad k_1\leq k_7\quad\text{and}\quad k_8\leq k_2\;,\quad\text{then}\quad k_2\leq k_5\;. \tag{38}$$

However, the opposite relationship is not always true,

$$\text{if}\quad\max\left\{4+\frac{r_1^2}{2}\;,\;\frac{5r_1^2-1}{2r_1-1}\right\}\leq r_1r_2+r_3\leq 2r_1^2\;,\quad\text{then}\quad\left\{\begin{array}{l}k_7\leq k_1,\;k_2\leq k_8\;,\\ k_2\leq k_5\;.\end{array}\right. \tag{39}$$

Find two other values k_9 and k_{10} where the Frobenius numbers $F(r_1r_2+r_1^2k_9,r_3-r_1^2k_9)$ and $F(r_1r_2+r_1^2k_{10},r_3-r_1^2k_{10})$ coincide with $\Phi(r_1,r_2,r_3)$,

$$k_9=\frac{r_3-r_1}{r_1^2}\;,\quad k_{10}=\frac{1-r_2}{r_1}\;. \tag{40}$$

Both of them correspond to the 2D symmetric semigroup $\mathsf{S}(r_1,r_3+r_1(r_2-1))$. In fact, by comparison with (27) we get $k_{10}=k_4$, so we have only one new special value $k=k_9$. By comparison the 1st formula in (40) and the 2nd formula in (18) we obtain,

$$k_9=\left\{\begin{array}{lll}>k_2 & \text{if} & r_1=2\\ =k_2 & \text{if} & r_1=3\\ <k_2 & \text{if} & r_1\geq 4\end{array}\right.\;. \tag{41}$$

4.2 Nonsymmetric semigroups $\mathsf{S}(r_1^2,r_1r_2+r_1^2k,r_3-r_1^2k)$

In this section we consider the case of numerical semigroups $\mathsf{S}(r_1^2,r_1r_2+r_1^2k,r_3-r_1^2k)$ with nonsymmetric representatives which were not observed in sequences with generating triples (4). This case is much more difficult to deal with by the reason explained in Remark 1: being 3-dimensional, nonsymmetric semigroup is generated by elements satisfying by Lemma 1,

$$r_3-r_1^2k\notin\mathsf{S}(r_1,r_2+r_1k)\;,\quad r_3-r_1^2k\leq F(r_1,r_2+r_1k)\;. \tag{42}$$

The 1st condition in (42) is necessary and sufficient, however the 2nd one is only necessary. Thus, the 2nd condition does not guarantee that the chosen k satisfies the 1st one. On the other hand, a straightforward application of the latter requirement is hard to perform.

There exists another problem which makes the construction of nonsymmetric semigroups with generating triples (15) not easy. Indeed, summarizing (19), (22), (30) and (34), the set $\Xi\subset\mathbb{Z}$ of the k-values, where nonsymmetric semigroups $\mathsf{S}(r_1^2,r_1r_2+r_1^2k,r_3-r_1^2k)$ can be observed, reads

$$\Xi:=\{k\mid\mu_1<k<\mu_2\}\;,\quad\mu_1=\max\{k_1,k_3,k_7\}\,,\quad\mu_2=\min\{k_2,k_5,k_8\}\;. \tag{43}$$

Thus, if a set Ξ is not empty then every $k_* \in \Xi$ is a candidate to make a semigroup with generating triple (15) nonsymmetric. However, what is remained open this is a question: does such k_* satisfy the 1st requirement in (42) ?

We can point out the definite values of k associated with nonsymmetric semigroups. For example, consider r_1, r_2, r_3 providing an integer $r_3 - r_1^2 k$ as a gap of semigroup $\mathsf{S}(r_1, r_2 + r_1 k)$,

$$r_3 - r_1^2 k_* = r_2 + r_1 k_* + 1 \;, \quad \gcd(r_1, r_2 + 1) = 1 \;. \tag{44}$$

Equation (44) has the following solution k_* which, by comparison with (32), is close to k_6,

$$k_* = \frac{r_3 - r_2 - 1}{r_1(r_1 + 1)} \;, \quad \gcd(r_1, r_3) = \gcd(r_1, r_2) = \gcd(r_1, r_2 + 1) = 1 \;. \tag{45}$$

Two last constraints in the right hand side of (45) forbid r_1 be divisible by 2. The claim $k_* \in \mathbb{Z}$ requires for r_2 and r_3 to be of distinct parities. It turns out that these properties suffice to give rise to infinite family of 2-parametric solutions. Below we give one of them,

$$r_1 = 2p - 1 \;, \quad r_2 = 4p - 1 \;, \quad r_3 = 2pk_*(2p - 1) + 4p \;, \quad p \in \mathbb{Z}_+ \;, \quad p \geq 2 \;. \tag{46}$$

In (46) the value of k_* can be taken on our choice. In Table 1 we give a numerical semigroup $\mathsf{S}(9, 21 + 9k, 80 - 9k)$ which has its nonsymmetric representatives for $k_* = 5, 6, 7$. In this conjunction, formulas (46) are corresponding to $k_* = 6$ and $p = 2$ while the other two values of k_* come not by (44), but via the other Diophantine equations of similar form.

4.3 Concluding Remarks

In this section we summarize the results on distribution of symmetric and nonsymmetric numerical semigroups $\mathsf{S}(r_1^2, r_1 r_2 + r_1^2 k, r_3 - r_1^2 k)$ governed by one parameter k running throughout the range of its special values k_i.

1. In the range $k_4 \leq k \leq k_2$ every $k \in \mathbb{Z}$ gives rise to the 2D or 3D one parametric numerical semigroups generated by the triple (15).

2. In the range $k_4 < k \leq k_3$ all numerical semigroups $\mathsf{S}(r_1^2, r_1 r_2 + r_1^2 k, r_3 - r_1^2 k)$ are symmetric and their minimal generating triple cannot be reduced. Their Frobenius numbers coincide with $\Phi(r_1, r_2, r_3)$ given by (16).

3. In the range $k_5 < k \leq k_2$, $k \neq k_6$, all numerical semigroups $\mathsf{S}(r_1^2, r_1 r_2 + r_1^2 k, r_3 - r_1^2 k)$ are generated by minimal pair $\{r_1^2, r_3 - r_1^2 k\}$ and therefore are symmetric. Their Frobenius numbers are distinct and differ from $\Phi(r_1, r_2, r_3)$.

4. There exist $k = k_4$ and $k = k_6$ such that the corresponding generating sets (15) of semigroups $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k)$ are reduced up to minimal pairs $\{r_1, r_3 - r_1^2k_4\}$ and $\{r_1^2, r_3 - r_1^2k_6\}$, respectively. Their Frobenius numbers coincide with $\Phi(r_1, r_2, r_3)$.

5. In the range $k_1 \le k \le k_7$ and $k_8 \le k \le k_2$, $k \ne k_9$, all numerical semigroups $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k)$ are generated by minimal pair $\{r_1r_2 + r_1^2k, r_3 - r_1^2k\}$. Their Frobenius numbers are distinct and differ from $\Phi(r_1, r_2, r_3)$.

6. There exists $k = k_9$ such that the corresponding generating set (15) is reduced up to minimal pair $\{r_1, r_3 + r_1r_2 - r_1)$, Its Frobenius number coincides with $\Phi(r_1, r_2, r_3)$.

7. In the range $\mu_1 < \varkappa < \mu_2$, $\varkappa \in \mathbb{Z}$, and $\varkappa \ne k_6$, $\varkappa \ne k_9$, numerical semigroups $\mathsf{S}(r_1^2, r_1r_2 + r_1^2\varkappa, r_3 - r_1^2\varkappa)$ admit their symmetric and nonsymmetric representatives, where μ_1, μ_2 are defined in (43).

In Table 1 we present the special values k_i of parameter k for two sequences of numerical semigroups discussed in [1] and for semigroup $\mathsf{S}(9, 21 + 9k, 80 - 9k)$. We give also the Frobenius numbers $F(\lfloor k_i \rfloor)$ associated with these semigroups for $k = \lfloor k_i \rfloor$, where $\lfloor u \rfloor$ denotes the floor function of u, i.e. $\lfloor u \rfloor$ gives the largest integer less than or equal to u.

Table 1. Semigroups and their Frobenius numbers.

	r_1, r_2, r_3	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9
$\mathsf{S}(4, 6 + 4k, 87 - 4k)$	2, 3, 87	-0.5	21	14.16	-1	15.68	14	-1.24	21.49	21.25
$F(\lfloor k_i \rfloor)$	Φ=89	89	5	89	89	77	89	-	5	5
$\mathsf{S}(9, 3 + 9k, 85 - 9k)$	3, 1, 85	0.33	9.11	5.66	0	8.25	7	0.21	9.32	9.11
$F(\lfloor k_i \rfloor)$	Φ=167	167	23	167	167	95	167	167	23	23
$\mathsf{S}(9, 21 + 9k, 80 - 9k)$	3, 7, 80	-1.66	8.55	4.53	-2	7.53	6.08	-2.21	8.76	8.55
$F(\lfloor k_i \rfloor)$	Φ=193	193	55	193	193	109	121	-	55	55

In accordance with item 7 of above summary, below we give the values of $\varkappa_i$, $\mu_1 < \varkappa_i < \mu_2$ and $\varkappa_i \in \mathbb{Z}$ together with Frobenius numbers $F(\varkappa_i)$ of corresponding semigroups,

$$\mathsf{S}(4, 6 + 4\varkappa, 87 - 4\varkappa) \;:\; \varkappa_1 = 15,\; F(15) = 77\;,$$

$$\mathsf{S}(9, 3 + 9\varkappa, 85 - 9\varkappa) \;:\; \varkappa_1 = 6,\; \varkappa_2 = 7,\; F(6) = F(7) = 167\;,\;\; \varkappa_3 = 8,\; F(8) = 95\;,$$

$$\mathsf{S}(9, 21 + 9\varkappa, 80 - 9\varkappa) \;:\; \varkappa_1 = 5,\; F(5) = 166\;,\;\; \varkappa_2 = 6,\; F(6) = 121\;,\;\; \varkappa_3 = 7,\; F(7) = 109\;.$$

The generating sets of all three semigroups are satisfied (25), i.e. $k_1 \le k_3 \le k_2$. Note that in the whole range of varying k-parameter, $k_1 \le k \le k_3 \le k_2$ and $k_1 \le k_3 \le k \le k_2$, including the

values $\varkappa_i$, both sequences of numerical semigroups $\mathsf{S}(4, 6+4k, 87-4k)$ and $\mathsf{S}(9, 3+9k, 85-9k)$ give rise only to symmetric semigroups either three-dimensional or two-dimensional. In contrast to them, the numerical semigroups $\mathsf{S}(9, 21+9\varkappa_i, 80-9\varkappa_i)$ for $\varkappa_i = 5, 6, 7$ are three-dimensional and nonsymmetric.

5 Symmetric semigroups in the range $k_1 \le k \le k_2 \le k_3$

In this section we give a detailed analysis on numerical semigroups $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k)$ with parameter k running in the intervals (24) where all semigroups are always symmetric and have the Frobenius number $\Phi(r_1, r_2, r_3)$ given by (16). Find r_1 such that two inequalities in the left and right hand sides in (24) imposed on $r_1r_2 + r_3$ become consistent,

$$3 + 2r_1 \le r_1r_2 + r_3 \le \frac{r_1^2 + 5r_1 - 3}{r_1 - 1} \quad \rightarrow \quad \frac{r_1(r_1-4)}{r_1 - 1} \le 0 \quad \rightarrow \quad 2 \le r_1 \le 4\ . \tag{47}$$

Estimate the total number N of such semigroups keeping in mind that according to (19) and (20) k is varying in interval $k_1 \le k \le k_2$,

$$\begin{aligned} N \ &\le\ \left\lfloor \frac{r_3 - 3}{r_1^2} \right\rfloor - \left\lfloor \frac{2 - r_2}{r_1} \right\rfloor + 1 \le \frac{r_3 - 3}{r_1^2} - \frac{2 - r_2}{r_1} + 2 = \frac{r_1r_2 + r_3 - (2r_1 + 3)}{r_1^2} + 2 \\ &\le\ \frac{1}{r_1^2}\left(\frac{r_1^2 + 5r_1 - 3}{r_1 - 1} - 2r_1 - 3\right) + 2 = \frac{4 - r_1}{r_1(r_1-1)} + 2 = \frac{(r_1-2)^2}{r_1(r_1-1)} + 1 < 2\ . \end{aligned} \tag{48}$$

Thus, a sequence of symmetric numerical semigroups $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k)$ is empty ($N = 0$) or contains only one semigroup ($N = 1$) for every choice of r_1, r_2, r_3. In order to find all k providing the case (24) we consider according to (47) all values of r_1 separately.

- $r_1 = 2$, a semigroup $\mathsf{S}(4, 2r_2 + 4k, r_3 - 4k)$, $2 \nmid r_2$, $2 \nmid r_3$.

  $$7 \le 2r_2 + r_3 \le 11\ , \quad \frac{2 - r_2}{2} \le k \le \frac{r_3 - 3}{4}\ . \tag{49}$$

  By recasting admitted equations $2r_2 + r_3 = e_2$ which satisfy the double inequality in the left hand side of (49) we have to omit those equations when $e_2 = 0 \pmod 2$, otherwise $2 \mid r_3$.

  $$\left\{\begin{array}{rcll} 2r_2 + r_3 &=& 7\ , & \text{has 2 solutions}: \{r_2 = 1, r_3 = 5\};\ \{r_2 = 3, r_3 = 1\} \\ 2r_2 + r_3 &=& 9\ , & \text{has 2 solutions}: \{r_2 = 1, r_3 = 7\};\ \{r_2 = 3, r_3 = 3\} \\ 2r_2 + r_3 &=& 11\ , & \text{has 3 solutions}: \{r_2 = 1, r_3 = 9\};\ \{r_2 = 3, r_3 = 5\};\ \{r_2 = 5, r_3 = 1\} \end{array}\right.$$

  Below we give the corresponding solutions for k and the Frobenius numbers of associated numerical semigroups $\mathsf{S}(4, 2r_2 + 4k, r_3 - 4k)$ if they exist, i.e. if $k \in \mathbb{Z}$. We consider two

different cases, $2k + r_2 \neq 1$ and $2k + r_2 = 1$, or, in other words, when k does satisfy the double inequality in the right hand side of (49) and does not satisfy it, respectively.

$$\begin{array}{lllll} r_2 = 1, & r_3 = 7, & k = 1, & 2k + r_2 \neq 1, & F(4,6,3) = F(3,4) = 5 \\ r_2 = 3, & r_3 = 3, & k = 0, & 2k + r_2 \neq 1, & F(4,6,3) = F(3,4) = 5 \\ r_2 = 1, & r_3 = 9, & k = 1, & 2k + r_2 \neq 1, & F(4,6,5) = 7 \\ r_2 = 3, & r_3 = 5, & k = 0, & 2k + r_2 \neq 1, & F(4,6,5) = 7 \\ r_2 = 5, & r_3 = 1, & k = -1, & 2k + r_2 \neq 1, & F(4,6,5) = 7 \end{array} \tag{50}$$

$$\begin{array}{lllll} r_2 = 1, & r_3 = 5, & k = 0, & 2k + r_2 = 1, & F(4,2,5) = F(2,5) = 3 \\ r_2 = 3, & r_3 = 1, & k = -1, & 2k + r_2 = 1, & F(4,2,5) = F(2,5) = 3 \\ r_2 = 1, & r_3 = 7, & k = 0, & 2k + r_2 = 1, & F(4,2,7) = F(2,7) = 5 \\ r_2 = 3, & r_3 = 3, & k = -1, & 2k + r_2 = 1, & F(4,2,7) = F(2,7) = 5 \\ r_2 = 1, & r_3 = 9, & k = 0, & 2k + r_2 = 1, & F(4,2,9) = F(2,9) = 7 \\ r_2 = 3, & r_3 = 5, & k = -1, & 2k + r_2 = 1, & F(4,2,9) = F(2,9) = 7 \\ r_2 = 5, & r_3 = 1, & k = -2, & 2k + r_2 = 1, & F(4,2,9) = F(2,9) = 7 \end{array} \tag{51}$$

- $r_1 = 3$, a semigroup $\mathsf{S}(9, 3r_2 + 9k, r_3 - 9k)$, $3 \nmid r_2$, $3 \nmid r_3$,

$$9 \le 3r_2 + r_3 \le \frac{21}{2}\ , \quad \frac{2 - r_2}{3} \le k \le \frac{r_3 - 3}{9}\ . \tag{52}$$

Omit equations $3r_2 + r_3 = e_3$ such that $e_3 = 0 \pmod 3$, otherwise $3 \mid r_3$. Thus, we have,

$$3r_2 + r_3 = 10\ , \qquad \text{has 2 solutions} : \{r_2 = 1, r_3 = 7\};\ \{r_2 = 2, r_3 = 4\}\ . \tag{53}$$

Similarly to the previous case we give the corresponding solutions for k, $k \in \mathbb{Z}$, and the Frobenius numbers of associated numerical semigroups $\mathsf{S}(9, 3r_2+9k, r_3-9k)$ in two different cases, $3k + r_2 \neq 1$ and $3k + r_2 = 1$.

$$\begin{array}{lllll} r_2 = 2, & r_3 = 4, & k = 0, & 3k + r_2 \neq 1, & F(9,6,4) = 11\ , \\ r_2 = 1, & r_3 = 7, & k = 0, & 3k + r_2 = 1, & F(9,3,7) = F(3,7) = 11. \end{array} \tag{54}$$

- $r_1 = 4$, a semigroup $\mathsf{S}(16, 4r_2 + 16k, r_3 - 16k)$, $2 \nmid r_2$, $2 \nmid r_3$,

$$4r_2 + r_3 = 11\ , \quad \frac{2 - r_2}{4} \le k \le \frac{r_3 - 3}{16}\ . \tag{55}$$

It turns out that in the case $4k + r_2 \neq 1$ an equation (55) has not an integer solution in k. Thus, the only numerical semigroup $\mathsf{S}(16, 4r_2+16k, r_3-16k)$ with corresponding Frobenius number reads,

$$r_2 = 1,\ \ r_3 = 7,\ \ k = 0,\ \ 4k + r_2 = 1,\ \ F(16,4,7) = F(4,7) = 17\ . \tag{56}$$

6 Symmetric semigroups $\mathsf{S}(r_1^2, r_3 - r_1^2 k)$ and enumeration of integer points in plane curve

In section 4.1 we have observed a phenomenon of reduction of a number of three minimal generators $\{r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k\}$ up to two due to the linear dependence between the 1st and the 2nd generators if $r_1k + r_2 = 1$. In this conjunction ask about the other ways of similar reduction of semigroup's dimension when the linear dependence arises in the rest of two pairs of generators separately, namely, between the 1st and the 3rd generators or between the 2nd and the 3rd generators.

Regarding the first pair of the 1st and the 3rd generators, it can be proven that by assumptions $\gcd(r_1, r_3) = 1$ and $r_3 \in \mathbb{Z}^+$ such linear dependence could not happen. Indeed, if such dependence holds, $r_3 - r_1^2k = cr_1^2$, $c \in \mathbb{Z}^+$, then r_3 is divisible by r_1^2 or vanishes that contradicts the above assumptions.

Regarding the second pair of the 2nd and the 3rd generators, their linear dependence

$$r_3 - r_1^2k = f \cdot (r_1r_2 + r_1^2k) \ , \quad f \in \mathbb{Z}^+ \ , \tag{57}$$

could not happen since it also contradicts the assumptions $\gcd(r_1, r_3) = 1$ and $r_3 \in \mathbb{Z}^+$.

Thus, consider the following linear dependence,

$$r_1r_2 + r_1^2k = g \cdot (r_3 - r_1^2k) \ , \quad g \in \mathbb{Z}^+ \ . \tag{58}$$

The quadratic Diophantine equation (58) describes an algebraic curve of degree 2 in the $k - g$ plane. The number of points with integer coordinate, $k \in \mathbb{Z}$ and $g \in \mathbb{Z}^+$, of this curve coincides with a number of solutions of the Diophantine equation (58). It can be solved completely by reduction it to the Pell equation and further calculation of continued fractions [6].

In this section we give necessary conditions to have the integer solutions, $k \in \mathbb{Z}$ and $g \in \mathbb{Z}^+$, of equation (58) and present two examples associated with Arnold's experiments showing how these requirements help to find all triples with linear dependence between the 2nd and the 3rd generators.

First, note that g is divisible by r_1 that follows by (58) and assumption $\gcd(r_1, r_3) = 1$. Denote $X = g + 1$, $Y = k$ and rewrite equation (58) as follows,

$$r_1^2Y - r_3 + \frac{r_1r_2 + r_3}{X} = 0 \ , \quad Y \in \mathbb{Z}, \ \ X \in \mathbb{Z}^+, \ \ X \geq 2 \ . \tag{59}$$

The Diophantine equation (59) is solvable iff X takes its value among divisors of $r_1r_2 + r_3$. Hence the next Lemma follows.

Lemma 2 *Let a numerical semigroup* $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k)$ *be given such that* $k \in \mathbb{Z}$, $r_1, r_2, r_3 \in \mathbb{Z}^+$ *and* $r_1 \geq 2$, $\gcd(r_1, r_2) = \gcd(r_1, r_3) = 1$. *If the linear dependence (58) holds for* $g = g_*$ *and* $k = k_*$ *then a semigroup is isomorphic to the 2D symmetric semigroup* $\mathsf{S}(r_1^2, r_3 - r_1^2k_*)$, *and* g_* *is divisible by* r_1, *and* $r_1r_2 + r_3$ *is divisible by* $g_* + 1$.

Denote by $Q(r_1, r_2, r_3)$ the total number of solutions of equation (59) and by $\sigma_0(n)$ the number of positive divisors $\delta_i(n)$ of integer n, where $i = 1, \ldots, \sigma_0(n)$. First, by $g = X - 1 \geq 1$ we have to exclude the minimal divisor $\delta_{min}(r_1r_2+r_3) = 1$ from possible solutions X of (59). Next, let, by way of contradiction, the maximal divisor $\delta_{max}(r_1r_2 + r_3) = r_1r_2 + r_3$ coincides with one of solutions X. Then substituting it into (59) we get the final triple: $\{r_1^2, r_1r_2 + r_3 - 1, 1\}$. The occurrence of unity in the minimal generating set, $1 \in \mathbf{d}^3$, makes the associated numerical semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$ free of gaps and equivalent to the whole set of nonnegative integers, $\mathsf{S}\left(\mathbf{d}^3\right) \equiv \mathbb{Z}^+ \cup \{0\}$. For such semigroup the Frobenius number does not exist.

Thus, there exist $Q(r_1, r_2, r_3)$ different sporadic values $k = k_*$ which suffice to reduce the dimension of numerical semigroups $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k)$ up to 2 and induce a bijective correspondence between symmetric semigroups, $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k_*, r_3 - r_1^2k_*) \leftrightarrow \mathsf{S}(r_1^2, r_3 - r_1^2k_*)$, with nonempty sets of gaps. Keeping in mind both values $\delta_{min}(r_1r_2 + r_3)$ and $\delta_{max}(r_1r_2 + r_3)$, we can give the lower and upper bounds for $Q(r_1, r_2, r_3)$,

$$0 \leq Q(r_1, r_2, r_3) \leq \sigma_0(r_1r_2 + r_3) - 2 \; . \tag{60}$$

By (60) and Lemma 2 we come to the other Corollaries related to the cases when $Q(r_1, r_2, r_3) = 0$.

Corollary 1 *Let a numerical semigroup* $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k)$ *be given as in Lemma 2 and* $r_1r_2 + r_3$ *is a prime number. Then one cannot choose* $g = g_*$ *and* $k = k_*$ *such that the linear dependence (58) does define the 2D semigroup* $\mathsf{S}(r_1^2, r_3 - r_1^2k_*)$ *with nonempty set of gaps.*

Corollary 2 *Let a numerical semigroup* $\mathsf{S}(r_1^2, r_1r_2 + r_1^2k, r_3 - r_1^2k)$ *be given as in Lemma 2 and* $r_1r_2 + r_3 = p^2$ *where* p *is a prime number. Then* $Q(r_1, r_2, r_3) = 0$ *if* $p - 1$ *is not divisible by* r_1.

Proof By (60) there is only one candidate for solutions of the Diophantine equation (59) and by Corollary 2 this is $X = p$. Substituting it into (59) we get

$$r_1^2Y = p(p-1) - r_1r_2 \; , \qquad \text{or} \qquad r_1(r_2 + r_1Y) = p(p-1) \; . \tag{61}$$

Note that $\gcd(r_1, p) = 1$, or, keeping in mind that p is a prime number, this is equivalent that r_1 is not divisible by p. Indeed, let, by way of contradiction, $r_1 = v \cdot p$, $v \in \mathbb{Z}^+$. Then r_3 is also

divisible by p since, by $r_1r_2+r_3=p^2$, we have $r_3=p(p-vr_2)$. However, this contradicts the assumptions $\gcd(r_1,r_3)=1$ and $r_3\in\mathbb{Z}^+$.

If the Diophantine equation (61) is solvable then $p-1$ is necessarily divisible by r_1. Thus, if $p-1$ is not divisible by r_1 then equation (61) is not solvable, i.e. $Q(r_1,r_2,r_3)=0$. $\square$

Corollary 2 gives the necessary but not sufficient conditions for equation (61) to be solvable. Indeed, let $p-1=u\cdot r_1$, $u\in\mathbb{Z}^+$. Substituting it into (61) we get $r_1Y=u\cdot p-r_2$. Thus, the solvability of the last Diophantine equation in Y presumes an additional divisibility relation.

In the following Examples 1 and 2 we present the phenomenon of reduction of the 3D semigroup's dimension up to 2 in two different sequences of semigroups generated by triples (4) and discussed in [1]. In both Examples we have underlined those divisors δ_i of $r_1r_2+r_3$ which give rise to sporadic 2D semigroups with corresponding k_* and g_*.

Example 1 $\{d_1,d_2,d_3\}=\{4,6+4k,87-4k\}$, $\{r_1,r_2,r_3\}=\{2,3,87\}$

$$r_1r_2+r_3=93\ ,\quad \sigma_0(93)=4\ ,\quad \delta_i(93)=1,\underline{3},\underline{31},93\ ,\quad Q(2,3,87)=2\ ,$$
$$r_1\mid\delta_i(93)-1\ :\ \ 2\mid 2,\ \ 2\mid 30,\ \ 2\mid 92\ ,$$
$$(k_{1*},g_{1*})=(14,2),\ \ \mathsf{S}\left(\mathbf{d}_1^3\right)=\mathsf{S}(4,62,31),\ \ F(4,62,31)=F(4,31)=89\ ,$$
$$(k_{2*},g_{2*})=(21,30),\ \ \mathsf{S}\left(\mathbf{d}_2^3\right)=\mathsf{S}(4,90,3),\ \ F(4,90,3)=F(4,3)=5\ .$$

Example 2 $\{d_1,d_2,d_3\}=\{9,3+9k,85-9k\}$, $\{r_1,r_2,r_3\}=\{3,1,85\}$

$$r_1r_2+r_3=88\ ,\quad \sigma_0(88)=8\ ,\quad \delta_i(88)=1,2,\underline{4},8,11,\underline{22},44,88\ ,\quad Q(3,1,85)=2\ ,$$
$$r_1\mid\delta_i(88)-1:\ \ 3\mid 3,\ \ 3\mid 21,\ \ 3\mid 87\ ,$$
$$r_1\nmid\delta_i(88)-1:\ \ 3\nmid 1,\ \ 3\nmid 7,\ \ 3\nmid 10,\ \ 3\nmid 43\ ,$$
$$(k_{1*},g_{1*})=(7,3),\ \ \mathsf{S}\left(\mathbf{d}_1^3\right)=\mathsf{S}(9,66,22),\ \ F(9,66,22)=F(9,22)=167\ ,$$
$$(k_{2*},g_{2*})=(9,21),\ \ \mathsf{S}\left(\mathbf{d}_2^3\right)=\mathsf{S}(9,84,4),\ \ F(9,84,4)=F(9,4)=23\ .$$

Regarding the 3rd semigroup $\mathsf{S}(9,21+9k,80-9k)$, where nonsymmetric representatives are admitted (see section 4.2), we have $r_1r_2+r_3=101$, $\sigma_0(101)=2$ that by Corollary 1 results in $Q(3,7,80)=0$, i.e. the Diophantine equation (59) has no solutions.

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